Automatic Verification of Probabilistic Programs

Joost-Pieter Katoen & **Kevin Batz**

Lecture on Probabilistic Programming

Automatic Verification of Probabilistic Loops

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Motivation:

Verification of **Randomized Algorithms** and **Stochastic Processes**

The Bounded Retransmission Protocol [Helmink *et al.* '93, D'Argenio *et al.* '97]

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Is the probability of failing to transmit file (reaching $fail = F$) at most 0.0001?

Is the expected number of retransmission tries ($totalFail$) at most 10?

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Sample uniformly from $\{0, \ldots, n-1\}$ using fair coin flips only.

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$v \coloneqq 1\,;\, c \coloneqq 0\,;\, \mathit{term} \coloneqq 0\,;$

`while` $(\,\mathit{term} = 0\,)\,\{$

$\quad v \coloneqq 2 \cdot v\,;$

$\quad \{\, c \coloneqq 2 \cdot c \,\}\ [1/2]\ \{\, c \coloneqq 2 \cdot c + 1 \,\}\,;$

$\quad$ `if` $(\,v \geq n\,)\,\{$

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$\quad\quad$ `else` $\{\, v \coloneqq v - n\,;\, c \coloneqq c - n \,\}$

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Is $\Pr(c = i) = 1/n$ for all $i \in \{0, \ldots, n-1\}$?

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What is the probability of reaching a state $fail = F$?

What is the termination probability?

Recap: Weakest Preexpectations & Loop Invariants

Consider the complete lattice $(\mathbb{E}, \sqsubseteq)$ of *expectations*:

$$\mathbb{E} = \{f \mid f \colon \mathsf{States} \to \mathbb{R}^{\infty}_{\geq 0}\}$$

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\mathsf{wp}[\![\texttt{while}\,(\,c=1\,)\,\{\,c \coloneqq 0\ [1/2]\ x \coloneqq x+1\,\}]\!]\,(x) &= [c=1] \cdot (x+1) + [c \neq 1] \cdot x
\end{aligned}$$

C	$\mathsf{wp}\,[\![C]\!]\,(f)$
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$C_1\,;\,C_2$	$\mathsf{wp}[\![C_1]\!]\,(\mathsf{wp}[\![C_2]\!]\,(f))$
$\{\,C_1\,\}\,[\,p\,]\,\{\,C_2\,\}$	$p \cdot \mathsf{wp}[\![C_1]\!]\,(f) + (1-p) \cdot \mathsf{wp}[\![C_2]\!]\,(f)$
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The lfp is taken w.r.t. $\sqsubseteq$ on expectations.

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$$
\begin{array}{l}
\texttt{if}\ (y = 5)\ \{ \\
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This does not hold for loops (Lec. 13). How to tackle the verification of loops?

Verifying Probabilistic Loops

... by means of quantitative loop invariants.

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Problem: for $\mathsf{wp}[\![\texttt{while}\,(\,\varphi\,)\,\{\,body\,\}]\!]\,(f) = \mathsf{lfp}\,\Phi_f$, where

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Solution: Quantitative Loop Invariants

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Theorem (Superinvariants for Loops)

Let $I \in \mathbb{E}$. We have

$$\underbrace{\Phi_f(I) \sqsubseteq I}_{I \text{ is wp-superinvariant}} \quad \textit{implies} \quad \mathsf{wp}[\![\texttt{while}\,(\varphi)\,\{\,body\,\}]\!]\,(f) \sqsubseteq I\,.$$

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Loop invariants reduce reasoning about loops to **reasoning about one loop iteration.**

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Goal: $\mathsf{wp}[\![\texttt{while}\,(c = 1)\,\{\,c \coloneqq 0\ [1/2]\ x \coloneqq x + 1\,\}]\!]\,(x)$

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Being able to decide $\Phi_f(I) \sqsubseteq I$ enables automatic verification of loops.

Automatic Verification of Loops

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... by restricting expectations f, I and loops $\texttt{while}\,(\,\varphi\,)\,\{\,body\,\}$ such that

the wp-superinvariant condition $\Phi_f(I) \sqsubseteq I$ is decidable.

Definition

The set LinExp of *piecewise linear expectations* consists of all expectations f of the form

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Definition

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The following loop is linear:

$$
\begin{array}{l}
\texttt{while}\,(\mathit{term} = 0)\,\{ \\
\quad v \coloneqq 2 \cdot v\,; \\
\quad \{\, c \coloneqq 2 \cdot c \,\}\,[1/2]\,\{\, c \coloneqq 2 \cdot c + 1 \,\}\,; \\
\quad \texttt{if}\,(v \geq n)\,\{ \\
\qquad \texttt{if}\,(c < n)\,\{\, \mathit{term} \coloneqq 1 \,\} \\
\qquad \texttt{else}\,\{\, v \coloneqq v - n\,;\, c \coloneqq c - n \,\} \\
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For $f, g \in \mathsf{LinExp}$, *it is decidable whether* $f \sqsubseteq g$ *holds.*

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Theorem

Let $f, I \in \mathsf{LinExp}$ *and let* `while` $(\varphi)\ \{\ body\ \}$ *be linear. Then* $\Phi_f(I) \sqsubseteq I$ *is decidable.*

Automatic Verification of Probabilistic Loops

Given: $f, g \in \mathsf{LinExp}$ with

$$f = [\varphi_1] \cdot e_1 + \ldots + [\varphi_n] \cdot e_n \quad \text{and} \quad g = [\psi_1] \cdot a_1 + \ldots + [\psi_m] \cdot a_m$$

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Idea: Compute logical formula Θ over Vars such that

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Given: $f, g \in \mathsf{LinExp}$ with

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The latter problem is decidable using **Satisfiability Modulo Theories** techniques.[a]

[a] `https://www.microsoft.com/en-us/research/project/z3-3/`

Hence, our example

$$\mathsf{wp}[\![\texttt{while}\,(c=1)\,\{\,c \coloneqq 0\,[1/2]\,x \coloneqq x+1\,\}]\!]\,(x) \;\sqsubseteq\; [c \neq 1]\cdot x + [c=1]\cdot(x+1)$$

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Question: What to do if $\Phi_f(I) \sqsubseteq I$ does **not** hold?

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Mitigate this problem using a more powerful proof rule: k-induction.

Improved Automatic Verification of Loops

... by means of k-induction.

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Given $h, h' \in \mathbb{E}$, define their *pointwise minimum* by $h \sqcap h' = \lambda\sigma.\, \min\{h(\sigma), h'(\sigma)\}$.

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For $f, I \in \mathbb{E}$ and loop $C = \texttt{while}\,(\,\varphi\,)\,\{\,body\,\}$, it holds:

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Theorem

For $f, I \in \mathsf{LinExp}$, linear C, and $k \geq 1$, the condition for k-induction is decidable.

Follows from the fact that for $h, h' \in \mathsf{LinExp}$, one can compute $h \sqcap h' \in \mathsf{LinExp}$.

Back to our example

$$\mathtt{wp}[\![\,\mathtt{while}\,(\,c=1\,)\,\{\,c \coloneqq 0\;[\,1/2\,]\;x \coloneqq x+1\,\}\,]\!]\,(x) \;\sqsubseteq\; x+1\,,$$

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k-induction is more powerful and decidable in the linear setting.

We have implemented k-induction for probabilistic loops:

KIPRO2: k-Induction for PRObabilistic PROgrams[1]

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For C_{brp} given by

$$\texttt{while}\,(\,sent < N \wedge fail < F\,)\,\{$$
$$\{\underbrace{fail := fail + 1\,;\, totalFail := totalFail + 1}_{\text{failed transmission}}\}[0.1]\{\underbrace{fail := 0\,;\, sent := sent + 1}_{\text{successful transmission}}\}\}$$

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We fully automatically verify that

$$\mathsf{wp}[\![C_{\text{brp}}]\!]\,(totalFail) \sqsubseteq [N \leq 3] \cdot (totalFail + 1) + [N > 3] \cdot \infty$$

by means of is 4-induction.

Optimal Discrete Uniform Generation from Coin Flips [Lumbroso '13]

Sample uniformly from $\{0, \ldots, n-1\}$ using fair coin flips only.

$v \coloneqq 1\,;\, c \coloneqq 0\,;\, term \coloneqq 0\,;$

`while` $(\, term = 0 \,)\, \{$

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For $n \in \{2,3,4,5\}$, we automatically verify

$\mathbf{Pr}(c = i) \leq 1/n$ **for all** $i \in \{0, \ldots, n-1\}$

by means of 2-,3-, and 5-induction.

k-induction is more powerful and decidable in the linear setting.

The property

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We have $\Phi_x(2x+1) \not\sqsubseteq 2x+1$, $\Phi_x((\Phi_x(2x+1) \sqcap 2x+1)) \not\sqsubseteq 2x+1, \ldots$

Solution: **Invariant Synthesis**

Invariant Synthesis

... i.e., computing superinvariants.

Invariant Synthesis

Idea: To verify

$$\mathsf{wp}[\![\texttt{while}\,(c = 1)\,\{\,c \coloneqq 0\,[1/2]\,x \coloneqq x + 1\,\}]\!]\,(x) \;\sqsubseteq\; 2x + 1$$

Invariant Synthesis

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we compute an expectation I such that

1. I is a superinvariant: $\Phi_x(I) \sqsubseteq I$

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we compute an expectation I such that

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2. I is not too large: $I \sqsubseteq 2x+1$.

Invariant Synthesis

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We then have

$$\mathsf{wp}[\![\texttt{while}\,(c=1)\,\{\,c \coloneqq 0\ [1/2]\ x \coloneqq x+1\,\}]\!]\,(x) \underbrace{\sqsubseteq}_{I \text{ is superinvariant}} I$$

Invariant Synthesis

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Question: **How to compute I?**

$$\mathtt{wp}[\![\mathtt{while}\,(c = 1)\,\{\,c \coloneqq 0\ [1/2]\ x \coloneqq x + 1\,\}]\!]\,(x) \;\sqsubseteq\; 2x + 1$$

Goal: Compute $I \in \mathsf{LinExp}$ such that $\Phi_x(I) \sqsubseteq I$ and $I \sqsubseteq 2x + 1$.

Invariant Synthesis

$$\texttt{wp}[\![\texttt{while}\,(c = 1)\,\{\,c \coloneqq 0\ [1/2]\ x \coloneqq x + 1\,\}]\!]\,(x) \;\sqsubseteq\; 2x + 1$$

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Idea: Set up a *template*

$$T \;=\; [c \neq 1] \cdot x + [c = 1] \cdot (\alpha \cdot c + \beta \cdot x + \gamma)$$

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$$T \ = \ [c \neq 1] \cdot x + [c = 1] \cdot (\alpha \cdot c + \beta \cdot x + \gamma)$$

New goal: Find appropriate $\alpha, \beta, \gamma \in \mathbb{Q}$.

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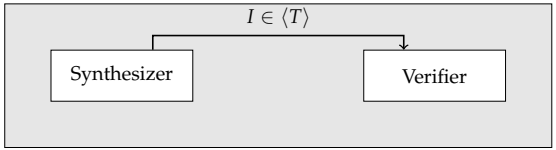

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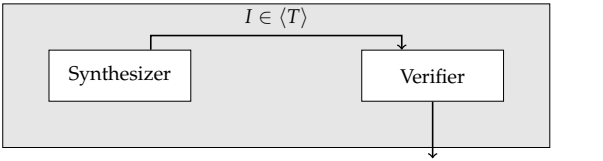

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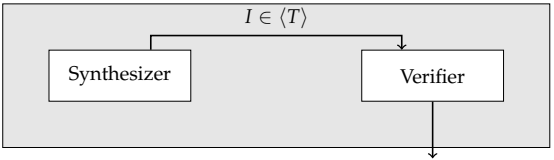

Remember:

$$\Phi_x(I) \sqsubseteq I \text{ iff } \forall \sigma\colon \Phi_x(I)(\sigma) \leq I(\sigma)$$

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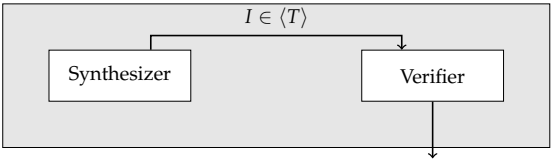

I is superinvariant and not too large ✓

Remember:

$$\Phi_x(I) \sqsubseteq I \text{ iff } \forall \sigma\colon \Phi_x(I)(\sigma) \leq I(\sigma)$$

$$I \sqsubseteq 2x+1 \text{ iff } \forall \sigma\colon I(\sigma) \leq 2 \cdot \sigma(x) + 1$$

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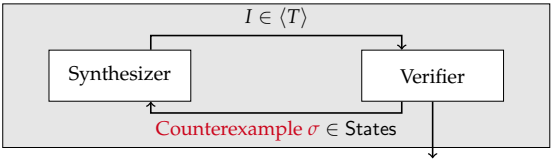

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Invariant Synthesis

$$\mathtt{wp}[\![\mathtt{while}\,(c=1)\,\{\,c \coloneqq 0\ [1/2]\ x \coloneqq x+1\,\}]\!]\,(x) \;\sqsubseteq\; 2x+1$$

$$T \;=\; [c \neq 1]\cdot x + [c=1]\cdot(\alpha\cdot c + \beta\cdot x + \gamma)$$

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Invariant Synthesis

$$\texttt{wp}[\![\texttt{while}\,(c=1)\,\{\,c \coloneqq 0\,[1/2]\,x \coloneqq x+1\,\}]\!](x) \;\sqsubseteq\; 2x+1$$

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Goal: Find $I \in \langle T \rangle$ such that $\Phi_x(I) \sqsubseteq I$ and $I \sqsubseteq 2x+1$.

Synthesizer: $I = [c=1]\cdot(0\cdot c + 0\cdot x + 0) + [c\neq 1]\cdot x$

Invariant Synthesis

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Verifier: cex. $\sigma(c)=1, \sigma(x)=1$ with $\Phi_x(I)(\sigma) > I(\sigma)$

Invariant Synthesis

$$\texttt{wp}[\![\texttt{while}\,(c=1)\,\{\,c \coloneqq 0\,[1/2]\,x \coloneqq x+1\,\}]\!]\,(x) \;\sqsubseteq\; 2x+1$$

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Rule out the cex. σ in all future iterations.

$$\texttt{wp}[\![\texttt{while}\,(c=1)\,\{\,c \coloneqq 0\,[1/2]\,x \coloneqq x+1\,\}]\!]\,(x) \;\sqsubseteq\; 2x+1$$

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$$\Phi_x(T) \quad = [c\neq 1]\cdot x + [c=1]\cdot\frac{1}{2}\cdot(\alpha\cdot c + \beta\cdot(x+1) + \gamma + x)$$

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$$\begin{aligned}\Phi_x(T) &= [c\neq 1]\cdot x + [c=1]\cdot\frac{1}{2}\cdot(\alpha\cdot c + \beta\cdot(x+1) + \gamma + x)\\ T &= [c\neq 1]\cdot x + [c=1]\cdot(\alpha\cdot c + \beta\cdot x + \gamma)\end{aligned}$$

Invariant Synthesis

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$$\Phi_x(T)(\sigma) = \frac{1}{2}\cdot(\alpha\cdot 1 + \beta\cdot(1+1) + \gamma + 1)$$

$$T(\sigma) = \alpha\cdot 1 + \beta\cdot 1 + \gamma$$

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$$T(\sigma) = \quad \alpha\cdot 1 + \beta\cdot 1 + \gamma$$

Learn constraint $\frac{1}{2}\cdot(\alpha\cdot 1 + \beta\cdot 2 + \gamma + 1) \leq \alpha\cdot 1 + \beta\cdot 1 + \gamma$

Invariant Synthesis

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$\frac{1}{2}(\alpha + 2\beta + \gamma + 1) \leq \alpha + \beta + \gamma$

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Invariant Synthesis

$$\texttt{wp}[\![\texttt{while}\,(c=1)\,\{\,c \coloneqq 0\,[1/2]\,x \coloneqq x+1\,\}]\!](x) \;\sqsubseteq\; 2x+1$$

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Verifier: cex. $\sigma(c)=1, \sigma(x)=2$ with $\Phi_x(I)(\sigma) > I(\sigma)$

Learned constraints:

$\frac{1}{2}(\alpha + 2\beta + \gamma + 1) \leq \alpha + \beta + \gamma$

Invariant Synthesis

$$\mathtt{wp}[\![\mathtt{while}\,(c=1)\,\{\,c \coloneqq 0\ [1/2]\ x \coloneqq x+1\,\}]\!]\,(x) \;\sqsubseteq\; 2x+1$$

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$$\Phi_x(T) \quad = [c\neq 1]\cdot x + [c=1]\cdot\frac{1}{2}\cdot(\alpha\cdot c + \beta\cdot(x+1) + \gamma + x)$$

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Invariant Synthesis

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Learned constraints:
$\frac{1}{2}(\alpha+2\beta+\gamma+1) \leq \alpha+\beta+\gamma$

$$\Phi_x(T)(\sigma) = \frac{1}{2}\cdot(\alpha\cdot 1 + \beta\cdot(2+1) + \gamma + 1)$$

$$T(\sigma) = \alpha\cdot 1 + \beta\cdot 2 + \gamma$$

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$\frac{1}{2}(\alpha+2\beta+\gamma+1) \leq \alpha+\beta+\gamma$

$$\Phi_x(T)(\sigma) = \frac{1}{2}\cdot(\alpha\cdot 1+\beta\cdot(2+1)+\gamma+1)$$

$$T(\sigma) = \alpha\cdot 1+\beta\cdot 2+\gamma$$

Learn constraint $\frac{1}{2}\cdot(\alpha\cdot 1+\beta\cdot 3+\gamma+1) \leq \alpha\cdot 1+\beta\cdot 2+\gamma$

Invariant Synthesis

$$\mathtt{wp}[\![\mathtt{while}\,(c=1)\,\{\,c \coloneqq 0\ [1/2]\ x \coloneqq x+1\,\}]\!]\,(x) \;\sqsubseteq\; 2x+1$$

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Goal: Find $I \in \langle T \rangle$ such that $\Phi_x(I) \sqsubseteq I$ and $I \sqsubseteq 2x+1$.

Learned constraints:
$\frac{1}{2}(\alpha + 2\beta + \gamma + 1) \leq \alpha + \beta + \gamma$
$\frac{1}{2}(\alpha + 3\beta + \gamma + 1) \leq \alpha + 2\beta + \gamma$

Invariant Synthesis

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Invariant Synthesis

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Goal: Find $I \in \langle T \rangle$ such that $\Phi_x(I) \sqsubseteq I$ and $I \sqsubseteq 2x+1$.

Synthesizer: $I = [c=1]\cdot(0\cdot c + 1\cdot x + 1) + [c\neq 1]\cdot x$

Verifier: verified! ✓

Learned constraints:

$\frac{1}{2}(\alpha + 2\beta + \gamma + 1) \leq \alpha + \beta + \gamma$

$\frac{1}{2}(\alpha + 3\beta + \gamma + 1) \leq \alpha + 2\beta + \gamma$

Invariant Synthesis

We have implemented invariant synthesis for probabilistic loops:

CEGISRO2: **CounterExample Guided Synthesis for PRObabilistic PROgrams**

Invariant Synthesis

We have implemented invariant synthesis for probabilistic loops:

CEGISRO2: CounterExample Guided Synthesis for PRObabilistic PROgrams

Given piecewise linear f, g and linear $\texttt{while}\,(\,\varphi\,)\,\{\,body\,\}$, cegispro2 aims to verify

$$\texttt{wp}[\![\texttt{while}\,(\,\varphi\,)\,\{\,body\,\}]\!]\,(f) \;\sqsubseteq\; g$$

by computing a piecewise linear expectation I with $\Phi_f(I) \sqsubseteq I$ and $I \sqsubseteq g$.

Invariant Synthesis

We have implemented invariant synthesis for probabilistic loops:

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by computing a piecewise linear expectation I with $\Phi_f(I) \sqsubseteq I$ and $I \sqsubseteq g$.
Using the same technique cegispro2 can also compute upper bounds on

$$\mathsf{ert}[\![\texttt{while}\,(\,\varphi\,)\,\{\,body\,\}]\!]\,(0)$$

by computing an ert-superinvariant.

Invariant synthesis enables verifying programs with gigantic state spaces.

For C_{brp} given by

$$\texttt{while}\left(sent < 8 \cdot 10^9 \wedge fail < 10\right)\{$$

$$\{\underbrace{fail := fail + 1\,;\, totalFail := totalFail + 1}_{\text{failed transmission}}\}[0.01]\{\underbrace{fail := 0\,;\, sent := sent + 1}_{\text{successful transmission}}\}\}$$

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We fully automatically verify that

$$\mathsf{wp}[\![C_{\text{brp}}]\!]\left([\mathit{fail}=10]\right) \;\sqsubseteq\; [\mathit{sent}=0 \land \mathit{fail}=0]\cdot 0.0001 + [\neg(N=0 \land \mathit{fail}=0)]\cdot\infty\,.$$

in 58 seconds using 106 counterexamples.

Invariant Synthesis

Invariant synthesis enables upper bounding **expected runtimes**.

Consider the program C modeling a race between a tortoise and a hare:

$$\texttt{while}\,(\,h \leq t\,)\,\{$$
$$\quad t := t + 1;$$
$$\quad \{\,\texttt{skip}\,\}\,[\,1/2\,]\,\{\,h := 1/11 \cdot [h+0\rangle + \ldots + 1/11 \cdot [h+10\rangle\,\}\,\}$$

[2] counting only the number of loop iterations

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$$
\begin{aligned}
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&\quad t := t + 1;\\
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\end{aligned}
$$

We fully automatically compute an ert-superinvariant[2] and obtain

$$\mathsf{ert}[\![C]\!]\,(0) \;\sqsubseteq\; [h \leq t] \cdot (2/3 \cdot t - 2/3 \cdot h + 13/3) + [h > t] \cdot 1$$

in 1.3 seconds using 32 counterexamples.

[2] counting only the number of loop iterations

Conclusion

The concepts of this lecture series provide the foundations for the
automatic verification of probabilistic programs.

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Are you interested in writing a Bachelor/Master Thesis? Contact us!

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