Automated Deductive Verification of Probabilistic Programs

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PhD Defense

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Software bugs cost tremendous amounts of money and have caused human deaths.

Probabilistic programs

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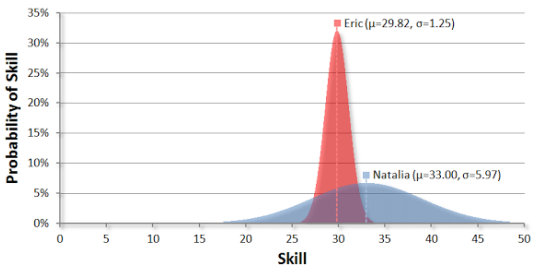
Machine Learning



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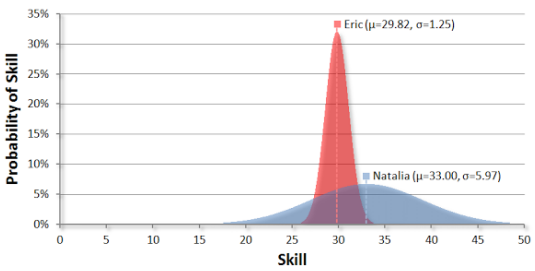

Security

The Bounded Retransmission Protocol [Helmink et al. '93, D'Argenio et al. '97]

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$$\textbf{Prove:} \quad \Pr[\Diamond\,(failed > M)] \leq 0.001$$

Outline & High-Level Goals

Outline:

The Weakest Preexpectation Calculus

Relatively Complete Verification [POPL '21]

Latticed k-Induction [CAV '21]

Inductive Synthesis of Inductive Invariants [TACAS '23]

Property Directed Reachability [CAV '20]

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High-Level Contributions:

1. Provide theoretical **foundations for automated deductive verifiers**
2. Advance the state-of-the-art for the **fully automatic verification of loops**

The Weakest Preexpectation Calculus

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What is the **probability** of reaching a given final state?

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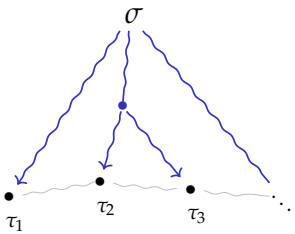

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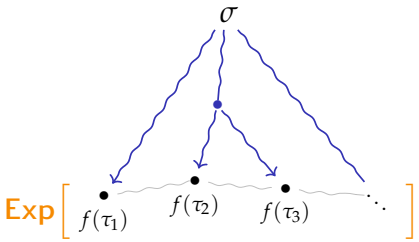

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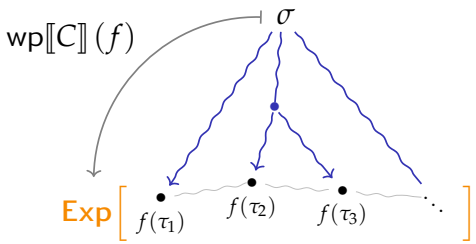

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$\{\,C_1\,\}\,[p]\,\{\,C_2\,\}$	$p \cdot \mathsf{wp}[\![C_1]\!]\,(f) + (1-p) \cdot \mathsf{wp}[\![C_2]\!]\,(f)$
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Verification of loops is tackled via **quantitative loop invariants.**

$\mathsf{wp}[\![\texttt{while}\,(\varphi)\,\{\,\mathit{body}\,\}]\!]\,(f) = \mathsf{lfp}\;\Phi_f$, where

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Theorem (Park Induction for Loops [Kaminski '19])

Let $I \in \mathbb{E}$. We have

$$\underbrace{\Phi_f(I) \sqsubseteq I}_{I \text{ is inductive loop invariant}} \qquad \textit{implies} \qquad \mathsf{wp}[\![\texttt{while}\,(\varphi)\,\{\,\mathit{body}\,\}]\!]\,(f) \sqsubseteq I\,.$$

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Loop invariants reduce reasoning about loops to **reasoning about one loop iteration.**

$$\mathsf{wp}[\![\texttt{while}\,(y = 1)\,\{\,y := 0\ [1/2]\ x := x + 1\,\}]\!]\,(x)$$

$$\mathsf{wp}[\![\texttt{while}\,(y=1)\,\{\,y:=0\ [1/2]\ x:=x+1\,\}]\!]\,(x)\ \sqsubseteq\ [y\neq 1]\cdot x+[y=1]\cdot(x+1)$$

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```
while (y = 1) {
    /// 1/2 · (x + [y ≠ 1] · (x + 1) + [y = 1] · (x + 2))
    { y := 0 } [1/2] { x := x + 1 }
    /// [y ≠ 1] · x + [y = 1] · (x + 1)
}
```

1. Compute $\Phi_x(I)$

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2. Check $\Phi_x(I)\sqsubseteq I$ via SMT

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What is a suitable syntax for expectations?

Relatively Complete Verification [POPL '21]

Goal: Provide a quantitative assertion language for deductive verifiers.

Ordinary Programs

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We provide a simple, yet expressive, language of expectations.

[1] [Clarke '76, Cook '78, Olderog '80]

[Flajolet, Pelletier, Soria '09]

$x := \mathsf{geometric}(1/4); y := \mathsf{geometric}(1/4);$

$\{t := x + y + 1\}\ [5/9]\ \{t := x + y\};$

$r := 1;$

`repeat 3 times {`

draw sequence of $2 \cdot t$ coin flips;

`if` (#heads $\neq$ #tails) $\{r := 0\};$

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Simple postexpectations can yield irrational, non-algebraic,... preexpectations

Arithmetic Expressions:

$$a \quad \longrightarrow \quad q \in \mathbb{Q}_{\geq 0} \mid \underbrace{x \in \mathsf{Vars}}_{\mathbb{Q}_{\geq 0}\text{-valued}} \mid a + a \mid a \cdot a \mid a \dot{-} a$$

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Theorem (POPL '21)

The language Exp *is expressive, i.e.,*

$$\forall \text{ programs } C \colon \ \forall f \in \mathsf{Exp} \colon \ \mathsf{wp}[\![C]\!](f) \in \mathsf{Exp}\,.$$

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Proof.

By induction on C. Most intricate part: loops.

Combines Gödel's β-function [Gödel 1931] with Dedekind cuts [Bertrand 1849]. □

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Exp is the basis of our modern automated deductive verifier CAESAR [OOPSLA '23].

Automatic Verification of Loops

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1. **Latticed k-Induction [CAV '21]**
2. **Inductive Synthesis of Inductive Invariants [TACAS '23]**
3. **Property Directed Reachability [CAV '20]**

Automatic Verification of Loops - The Challenge

$\mathsf{wp}[\![\texttt{while}\,(\varphi)\,\{\,body\,\}]\!]\,(f) = \mathsf{lfp}\;\Phi_f$, where

$$\Phi_f \colon \mathbb{E} \to \mathbb{E}, \qquad \Phi_f(h) = [\neg\varphi] \cdot f + [\varphi] \cdot \mathsf{wp}[\![body]\!]\,(h)$$

Theorem (Park Induction for Loops [Kaminski '19])

Let $I \in \mathbb{E}$. We have

$$\underbrace{\Phi_f(I) \sqsubseteq I}_{I \text{ is inductive loop invariant}} \qquad \textit{implies} \qquad \mathsf{wp}[\![\texttt{while}\,(\varphi)\,\{\,body\,\}]\!]\,(f) \sqsubseteq I\,.$$

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Problem: The converse direction does not hold in general!

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Leveraged by:

- [Winkler & Katoen '23]: Probabilistic Pushdown Automata
- [Yang et al. '24]: Static Analysis of Probabilistic Programs

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Corollary (k-Induction for Probabilistic Programs [CAV '21])

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k-inductivity is decidable for linear loop and piecewise linear expectations.

The Bounded Retransmission Protocol [Helmink et al. '93, D'Argenio et al. '97]

Goal: Send file of N packets via lossy channel.

Transmitting a packet fails with probability 0.1.

Allow for at most M retransmissions per packet.

$$\texttt{while}\,(\mathit{sent} < N \land \mathit{failed} \leq M)\,\{$$
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Our tool proves

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fully automatically by means of 4-induction.

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Solution: Invariant synthesis

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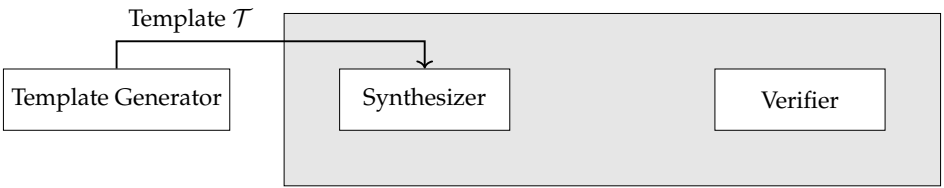

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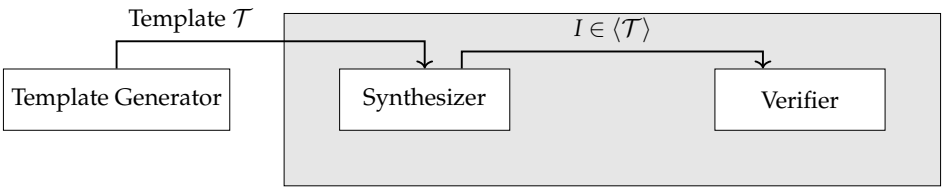

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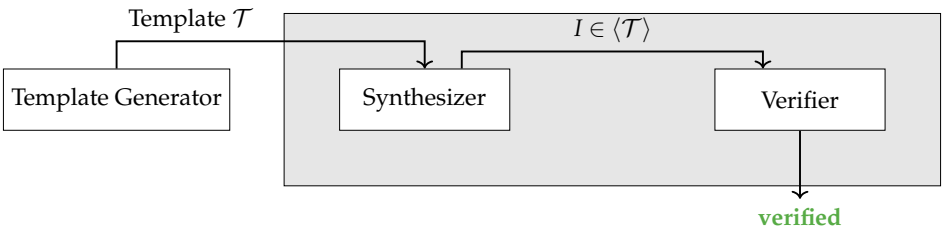

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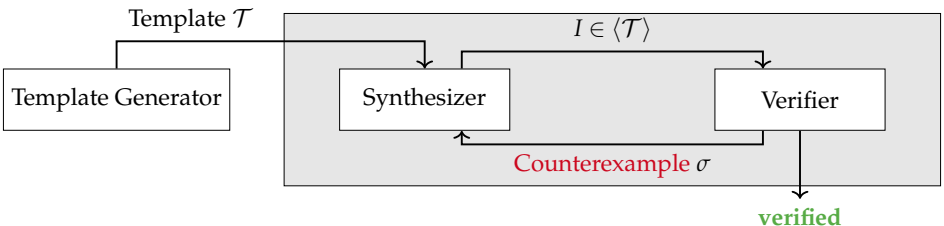

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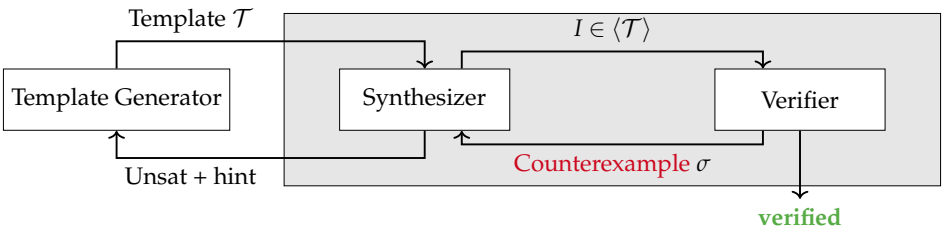

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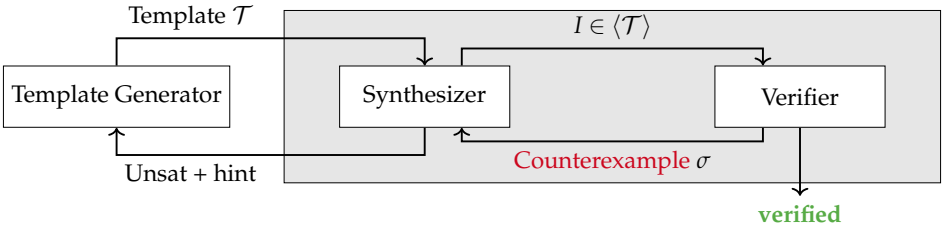

$$\mathcal{T} = [y = 0 \land x < 3] \cdot (\alpha_1 \cdot x + \beta_1 \cdot y + \gamma_1) + [y = 0 \land x \geq 3 \land x \leq 6] \cdot (\alpha_2 \cdot x + \beta_2 \cdot y + \gamma_2)$$

The Bounded Retransmission Protocol [Helmink et al. '93, D'Argenio et al. '97]

Goal: Send file of $N = 8 \cdot 10^9$ packets via lossy channel.

Transmitting a packet fails with probability 0.01.

Allow for at most $M = 10$ retransmissions per packet.

$$sent := 0; failed := 0;$$

$$\texttt{while}\,(sent < N \wedge failed \leq M)\,\texttt{do}$$

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Our tool proves $\mathsf{wp}[\![C]\!]\,([\mathit{failed} > M]) \sqsubseteq 0.001$ in **11** seconds.

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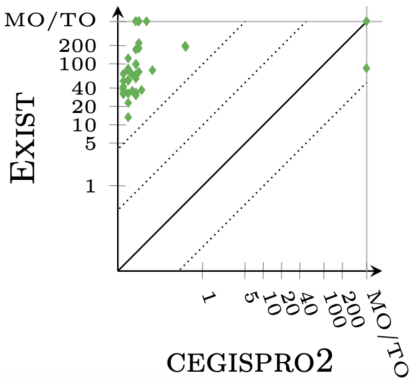
Automatic deductive program verification complements probabilistic model checking.

Mostly **infinite-state programs** from [Bao et al. '22] (TO=5min, MO=8gb)



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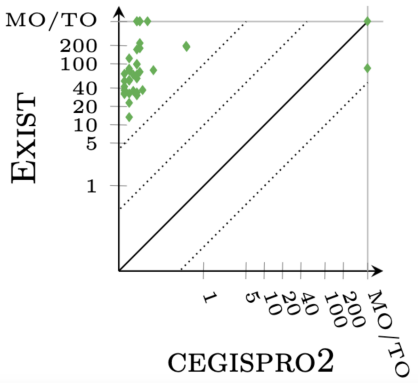

Our approach is competitive with the state-of-the-art invariant synthesis techniques.

Property Directed Reachability [CAV '20]

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- **Recent advances by [Kori et al. '22,'23]. Future work: Join forces!**

Scientific Contributions

1. **Computing Sampling Times [ESOP '18]**
2. **Quantitative Separation Logic [POPL '19]**
3. **Property Directed Reachability [CAV '20]**
4. **Generating Functions for PPs [LOPSTR '21]**
5. **Relatively Complete Verification [POPL '21]**
6. **Latticed k-Induction [CAV '21]**
7. **Entailment Checking in QSL [ESOP '22]**
8. **Weighted Programming [OOPSLA '22]**
9. **Amortized Expected Runtimes [POPL '23]**
10. **Invariant Synthesis [TACAS '23]**
11. **Verification Infrastructure for PPs [OOPSLA '23]**
12. **Programmatic Strategy Synthesis [POPL '24]**
13. **J.-P.: MDP. FP. PP [Festschrift JPK]**
14. **Verifying Continuous Programs [under sub.]**
15. **Quant. Quantifier Elimination [under sub.]**

Conclusion

- **Relatively Complete Verification [POPL '21]**
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Thank you!

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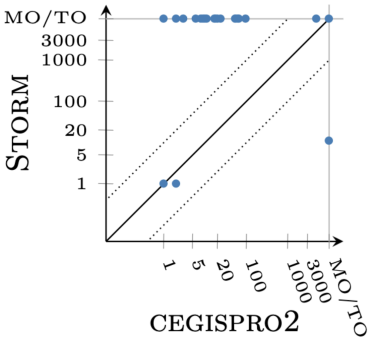
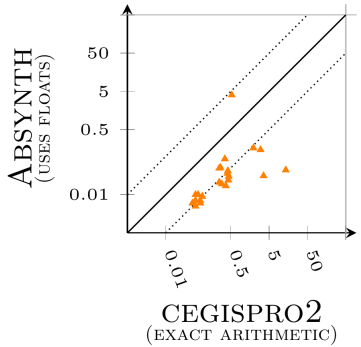
Thank you!
Arigato gozaimasu!

Finite-State Programs (Runtimes in seconds. TO=2h, MO=8gb)



Invariant synthesis can advance the state-of-the-art in probabilistic model checking.

Upper-bounding expected runtimes Programs from [Ngo et al. '18] (TO=5min, MO=8gb)



***cegispro2* can compute ERTs of infinite-state programs.**
Bounds are of similar quality. No floating-point issues.

Backup

		STORM			CEGISPRO2										
					INDUCT.-GUIDED				STATIC				DYNAMIC			
Prog	$\|S\|$	SP	DD	BEST	$\|S'\|$	$\|I\|$	t_s%	t	$\|S'\|$	$\|I\|$	t_s%	t	$\|S'\|$	$\|I\|$	t_s%	t
boundedrwmultistep	$1 \cdot 10^5$	MO	TO	3	33	10	40	3	–	–	–	TO	–	–	–	TO
		MO	TO	10	55	16	36	10	–	–	–	TO	–	–	–	TO
		MO	TO	–	–	–	–	TO	–	–	–	TO	–	–	–	TO
brp	$1 \cdot 10^{10}$	MO	TO	11	56	23	40	11	70	10	35	18	–	–	–	TO
		MO	TO	54	138	42	63	253	125	17	27	54	–	–	–	TO
		MO	TO	56	104	41	54	111	122	17	30	56	–	–	–	TO
brpfinitefamily	$16 \cdot 10^{13}$	TO	TO	8	53	7	72	10	67	7	44	8	54	9	52	29
		TO	TO	17	64	13	74	17	215	19	66	373	–	–	–	TO
		TO	TO	18	68	12	68	18	231	19	80	731	–	–	–	TO