Latticed k-Induction with an Application to Probabilistic Programs

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k-Induction [Sheeran et al. 2000]

- SAT-based technique for verifying (invariant properties of) finite transition systems

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Yes. Enables *fully automatic* verification of non-trivial properties.

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is not inductive but 2-*inductive*.

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Goal: Prove that P covers all reachable states of TS

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For verifying probabilistic programs, we have to ...

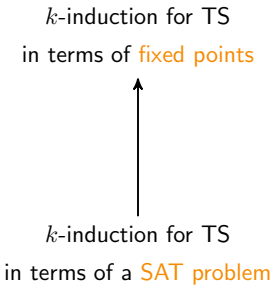
- ... leave the Boolean domain and reason about quantities
- ... reason about sets of paths rather than individual paths

Latticed k-Induction

k-induction for TS

in terms of a SAT problem

Latticed k-Induction



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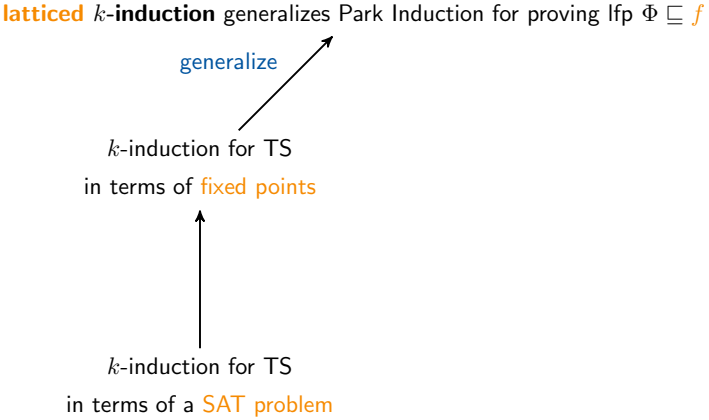

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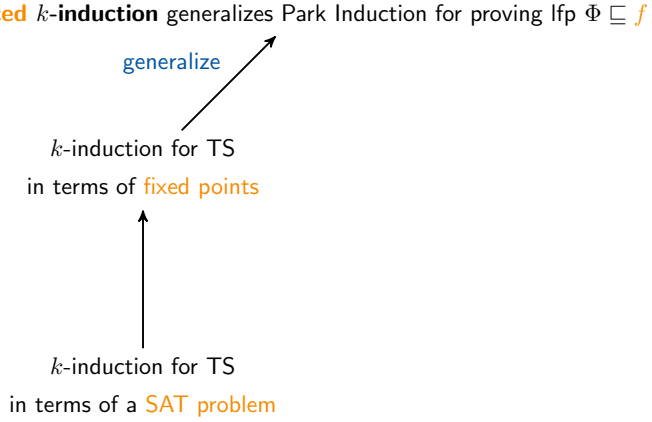

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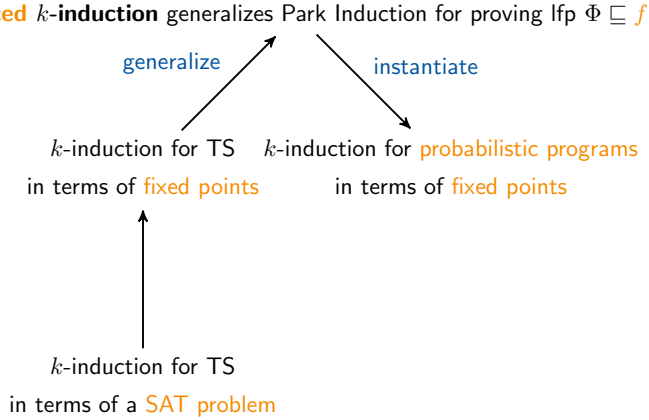

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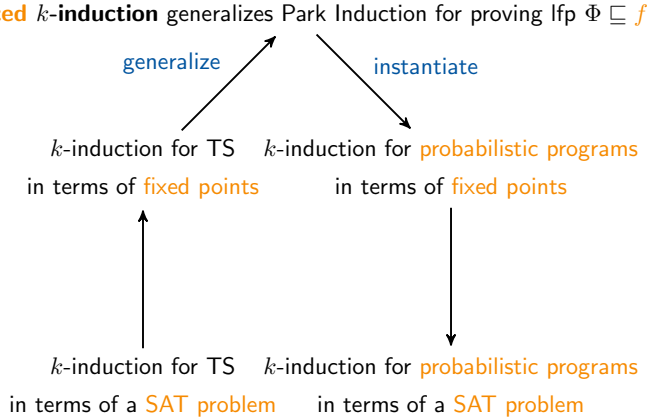

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Park Induction aka 1-induction:

$$\Phi(f) \sqsubseteq f \qquad \text{implies} \qquad \text{lfp } \Phi \sqsubseteq f$$

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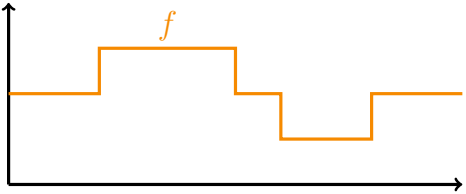

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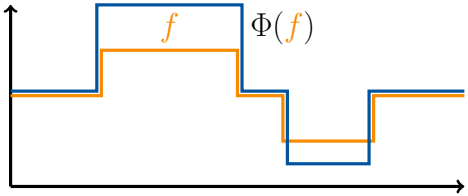

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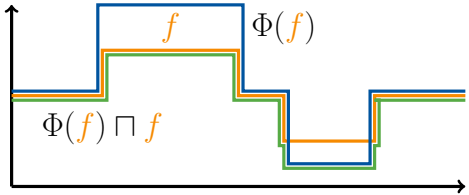

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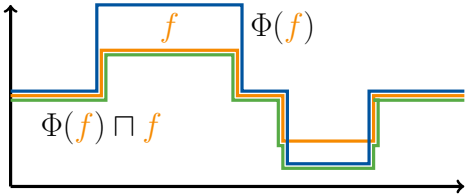

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Theorem (Latticed k-Induction)

For every $k \geq 1$,

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We call such f k-inductive invariant.

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Can be generalized to transfinite κ-induction (not in this talk).

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Theorem (Park Induction from k-Induction)

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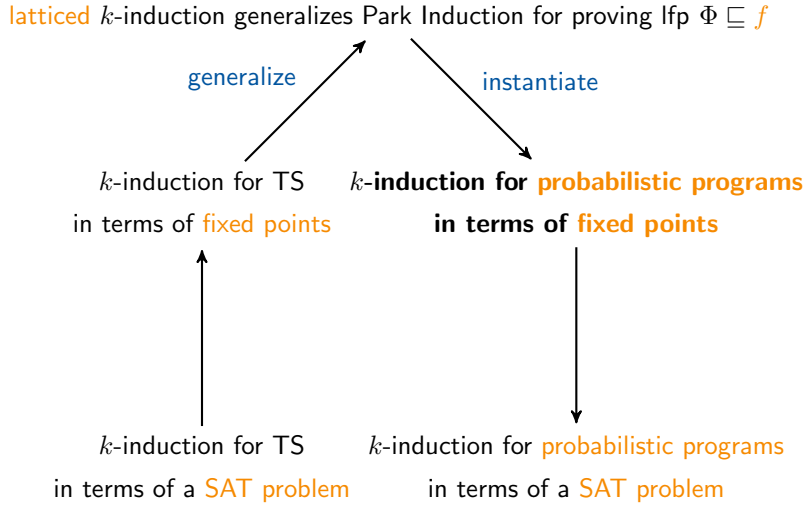
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$$\mathbb{E} = \{f \mid f \colon \Sigma \to \mathbb{R}^{\infty}_{\geq 0}\} \qquad \text{with} \qquad f \leq g \quad \text{iff} \quad \forall \sigma \in \Sigma \colon f(\sigma) \leq g(\sigma)$$

Weakest preexpectation transformer [Kozen'83, McIver'99, McIver & Morgan'05]:

$$\mathsf{wp}[\![C]\!] \colon \mathbb{E} \to \mathbb{E} \qquad \mathsf{wp}[\![C]\!]\,(g)\,(\sigma) \triangleq \text{expected value of } g \text{ evaluated in final states reached after executing } C \text{ on } \sigma$$

$$\begin{aligned}
\mathsf{wp}[\![x \coloneqq 5]\!]\,(x) &= 5 \\
\mathsf{wp}[\![\{\,\texttt{skip}\,\}\ [1/2]\ \{\,x \coloneqq x+2\,\}]\!]\,(x) &= \frac{1}{2}\cdot x + \frac{1}{2}\cdot(x+2) = x+1 \\
\mathsf{wp}[\![\{\,\texttt{skip}\,\}\ [1/2]\ \{\,x \coloneqq x+2\,\}]\!]\,([x=4]) &= \frac{1}{2}\cdot[x=4] + \frac{1}{2}\cdot[x=2] \\
\mathsf{wp}[\![\texttt{while}\,(c=1)\,\{\,c \coloneqq 0\ [1/2]\ x \coloneqq x+1\,\}]\!]\,(x) &= [c=1]\cdot(x+1) + [c\neq 1]\cdot x
\end{aligned}$$

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k-Induction for Probabilistic Programs

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Hence, latticed k-induction applies:

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For every $k \geq 1$,

$$\Phi\left(\Psi_f^{k-1}(f)\right) \leq f \quad \textit{implies} \quad \mathit{wp}[\![C]\!]\,(g) \leq f\ .$$

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Here

$$\Psi_f(h) = \Phi\,(h) \sqcap f \qquad \text{where for } h, h' \in \mathbb{E}, \quad h \sqcap h' = \lambda\sigma.\ \min\{h(\sigma), h'(\sigma)\}\ .$$

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Given *linear* $C = \texttt{while}\,(\,\varphi\,)\,\{\,C'\,\}$ and *piecewise linear* f, g, kipro2 *semi-decides* by SMT:

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If $\mathsf{wp}[\![C]\!]\,(g) \not\leq f$, KIPRO2 finds via *bounded model checking* some $\sigma \in \Sigma$ with

$$\mathsf{wp}[\![C]\!]\,(g)\,(\sigma) > f(\sigma)\ .$$

For C given by

$$\texttt{while}\,(\,c = 1\,)\,\{\,c \coloneqq 0\ [1/2]\ x \coloneqq x + 1\,\}\ ,$$

the property

$$\forall \text{ initial states } \sigma\colon \quad \mathsf{wp}[\![C]\!]\,(x)\,(\sigma) \leq \sigma(x) + 1$$

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also hold? No; counterexample by BMC: $\sigma(c) = 1, \sigma(x) = 6$.

Sampling uniformly from $\{elow, \ldots, ehigh\}$ using fair coin flips only [Lumbroso 2013]:

```
while(running = 0){

  v := 2*v;
  {c := 2*c+1}[0.5]{c := 2*c};
  if((not (v<n))){
    if((not (n=c)) & (not (n<c))){ # terminate
      running := 1
    }{
      v := v-n;
      c := c-n;
    }
  }{
    skip
  }

  # On termination, determine correct index
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For *arbitrary* array of fixed size $n = \{2, 3, 4, 5\}$, we verify

$$\Pr(\text{“sample fixed element”}) \leq \frac{1}{n} .$$

Table 2: Empirical results for the first benchmark set (time in seconds).

	postexpectation	variant	result	k	#formulae	formulae_t	sat_t	total_t
geo	c	1	ind	2	18	0.01	0.00	0.08
		2	ref	11	103	0.04	0.01	0.09
		3	ref	46	1223	0.39	0.04	0.48
unif_gen	$[c = i]$	1	ind	2	267	0.27	0.02	0.56
		2	ind	3	1402	1.45	0.10	1.81
		3	ind	3	1402	1.48	0.11	1.86
		4	ind	5	40568	47.31	15.70	63.28
		5	TO	–	–	–	–	–

...

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Thank you!